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ZETA FUNCTIONS FOR ANALYTIC MAPPINGS, LOG-PRINCIPALIZATION OF IDEALS, AND NEWTON POLYHEDRA

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Abstract. In this paper we provide a geometric description of the possible poles of the Igusa local zeta function $Z_{\Phi}(s,\mathbf{f})$ associated to an analytic mapping $\mathbf{f}=(f_1,\ldots,f_l):U(\subseteq K^n)\rightarrow K^l$, and a locally constant function Φ, with support in U, in terms of a log-principalizaton of the $K[x]-$ideal $\mathcal{I}_{\mathbf{f}}=(f_1,\ldots,f_l)$. Typically our new method provides a much shorter list of possible poles compared with the previous methods. We determine the largest real part of the poles of the Igusa zeta function, and then as a corollary, we obtain an asymptotic estimation for the number of solutions of an arbitrary system of polynomial congruences in terms of the log-canonical threshold of the subscheme given by $\mathcal{I}_{\mathbf{f}}$. We associate to an analytic mapping $\boldsymbol{f}=(f_1,\ldots,f_l)$ a Newton polyhedron $\Gamma(\boldsymbol{f})$ and a new notion of non-degeneracy with respect to $\Gamma(\boldsymbol{f})$. The novelty of this notion resides in the fact that it depends on one Newton polyhedron, and Khovanskii's non-degeneracy notion depends on the Newton polyhedra of $f_1,\ldots,f_l$. By constructing a log-principalization, we give an explicit list for the possible poles of $Z_{\Phi}(s,\mathbf{f})$, $l\geq 1$, in the case in which $\mathbf{f}$ is non-degenerate with respect to $\Gamma(\boldsymbol{f})$.

1. Introduction

Let K be a $p-$adic field, i.e. $[K:\mathbb{Q}_p]<\infty$. Let R_K be the valuation ring of K, P_K the maximal ideal of R_K, and $\overline{K}=R_K/P_K$ the residue field of K. The cardinality of the residue field of K is denoted by q, thus $\overline{K}=\mathbb{F}_q$. For $z\in K$, $ord(z)\in\mathbb{Z}\cup\{+\infty\}$ denotes the valuation of z, and $|z|_K=q^{-ord(z)}$ its absolute value. The absolute value $|\cdot|_K$ can be extended to K^l by defining $\|z\|_K=\max_{1\leq i\leq l}|z_i|_K$, for $z=(z_1,\ldots,z_l)\in K^l$.

Let $f_1,\ldots,f_l$ be polynomials in $K[x_1,\ldots,x_n]$, or, more generally, $K-$analytic functions on an open set $U\subset K^n$. We consider the mapping $\boldsymbol{f}=(f_1,\ldots,f_l):K^n\rightarrow K^l$, respectively, $U\rightarrow K^l$. Let $\Phi:K^n\rightarrow\mathbb{C}$ be a Schwartz-Bruhat function (with support in U in the second case). The Igusa local zeta function associated to

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the above data is defined as

$$Z_{\Phi}(s,\boldsymbol{f})=Z_{\Phi}(s,\boldsymbol{f},K)=\int\limits_{K^{n}}\Phi\left(x\right)\left\Vert \boldsymbol{f}(x)\right\Vert _{K}^{s}\mid dx\mid,$$

for $s\in\mathbb{C}$ with $\mathrm{Re}(s)>0$, where $\mid dx\mid$ is the Haar measure on K^{n} normalized in such a way that R_{K}^{n} has measure 1. We write $Z(s,\boldsymbol{f})$, $Z_{0}(s,\boldsymbol{f})$ and $Z_{W}(s,\boldsymbol{f})$ when Φ is the characteristic function of R_{K}^{n}, P_{K}^{n}, and an open compact subset W of K^{n}, respectively.

The function $Z_{\Phi}(s,\boldsymbol{f})$ admits a meromorphic continuation to the complex plane as a rational function of q^{-s}. Igusa established this result in the hypersurface case using Hironaka's resolution theorem [16, Theorem 8.2.1]. In the case $l\geq1$ the rationality of $Z_{\Phi}(s,\boldsymbol{f})$ was established by Meuser in [24], however, as mentioned in the review MR 83g:12015 of [24], a trick by Serre allows to deduce the general case from the hypersurface case. Denef gave a completely different proof of the rationality of $Z_{\Phi}(s,\boldsymbol{f})$, $l\geq1$, using p-adic cell decomposition [4]. The mentioned results do not give any information about the poles of $Z_{\Phi}(s,\boldsymbol{f})$ in the case $l>1$. In [37] the second author showed that a list of possible poles of $Z_{\Phi}(s,\boldsymbol{f})$, $l\geq1$, can be computed from an embedded resolution of singularities of the divisor $\cup_{i=1}^{l}f_{i}^{-1}(0)$ by using toroidal geometry. In the special case in which $\boldsymbol{f}$ is a non-degenerate homogeneous polynomial mapping the possible poles of $Z_{\Phi}(s,\boldsymbol{f})$ are given in [38].

In this paper we provide a geometric description of the possible poles of $Z_{\Phi}(s,\boldsymbol{f})$, $l\geq1$, in terms of a log-principalization of the $K\left[x\right]-$ideal $\mathcal{I}_{\boldsymbol{f}}=(f_{1},\ldots,f_{l})$ (see Theorem 2.4). At this point it is important to mention that the main result in [37] gives an algorithm to compute a list of possible poles of $Z_{\Phi}(s,\boldsymbol{f})$, $l\geq1$, in terms of an embedded resolution of singularities of the divisor $\cup_{i=1}^{l}f_{i}^{-1}(0)$, while Theorem 2.4 gives a list of candidates to poles in terms of a log-principalization of the ideal $\mathcal{I}_{\boldsymbol{f}}$. Typically our new method provides a much shorter list of possible poles (see Example 2.5). It is important to mention that in the case $l=1$ the problem of determining the poles of the meromorphic continuation of $Z_{\Phi}(s,\boldsymbol{f})$ in $\mathrm{Re}(s)<0$ has been studied extensively (see e.g. [3], [14], [28], [23], [32], [34]). The relevance of this problem is due to the existence of several conjectures relating the poles of $Z_{\Phi}(s,\boldsymbol{f})$ with the structure of the singular locus of $\boldsymbol{f}$. In the case of polynomials in two variables, as a consequence of the works of Igusa, Strauss, Meuser and the first author, there is a complete solution of this problem [14], [27], [23], [33]. For general polynomials the problem of determination of the poles of $Z_{\Phi}(s,\boldsymbol{f})$ is still open. There exists a generic class of polynomials named non-degenerate with respect to its Newton polyhedron for which it is possible to give a small set of candidates for the poles of $Z_{\Phi}(s,\boldsymbol{f})$. The poles of the local zeta functions attached to non-degenerate polynomials can be described in terms of Newton polyhedra. The case of two variables was studied by Lichtin and Meuser [21]. In [5], Denef gave a procedure based on monomial changes of variables to determine a small set of candidates for the poles of $Z_{\Phi}(s,\boldsymbol{f})$ in terms of the Newton polyhedron of f. This result was obtained by the second author, using an approach based on the $p-$adic stationary phase formula and Néron $p-$desingularization, for polynomials with coefficients in a non-archimedean local field of arbitrary characteristic [36], (see also [7], [29]).

In the case $l=1$, among the conjectures relating the poles of Igusa's zeta function with topology and singularity theory, we mention here a conjecture of Igusa that proposes that the real parts of the poles of the Igusa zeta function of $\boldsymbol{f}$ are roots

of the Bernstein polynomial of $\boldsymbol{f}$ (see e.g. [3], [16], and references therein). It seems reasonable to believe that such relations between poles and singularity theory extend to the case $l>1$. Indeed, recently it was proved that the above-mentioned conjecture of Igusa is valid in the case in which $\mathcal{I}_{\boldsymbol{f}}$ is a monomial ideal [13].

In the case $l=1$, the largest real part of the poles of the Igusa zeta function has been extensively studied both in the archimedean and non-archimedean cases [7], [21], [31], [36]. In the case $l\geq 1$ we show that the largest real part $-\lambda\left(\mathcal{I}_{\boldsymbol{f}}\right)$ of the poles of the Igusa zeta function attached to $\boldsymbol{f}$ can easily be determined from a log-principalization of the ideal $\mathcal{I}_{\boldsymbol{f}}$ (see Theorem 2.7). As a consequence of this result we obtain an asymptotic estimation for the number of solutions of an arbitrary system of polynomial congruences in terms of the log-canonical threshold of a log-principalization (see Corollary 2.9, and the comments that follow). At this point we have to mention that in the case $l=1$ Loeser found lower and upper bounds for $\lambda\left(\mathcal{I}_{\boldsymbol{f}}\right)$ in terms of certain geometric invariants introduced by Teissier [22, Theorem 2.6 and Proposition 3.1.1], [30]. In this form he derived a geometric bound for the number of solutions of a polynomial congruence involving one polynomial.

If $\boldsymbol{f}$ is a polynomial mapping with coefficients in a number field F, then for every maximal ideal P of the ring of algebraic integers of F, we can consider $Z(s,\boldsymbol{f},K)$, $l\geq 1$, where K is the completion of F with respect to P. We give an explicit formula for $Z(s,\boldsymbol{f},K)$, $l\geq 1$, that is valid for almost all P (see Theorem 2.10). The proof of this formula follows by adapting the argument given by Denef for the case $l=1$ [6].

One can also associate to a sheaf of ideals $\mathcal{I}$ on a smooth algebraic variety (over a field of characteristic zero) a motivic zeta function (see Definition 2.16). By using a log-principalization of $\mathcal{I}$ we give a similar explicit formula for it (see Theorem 2.17). The proof is a reasonably straightforward generalization of the one given by Denef and Loeser in [8]. By specializing to Euler characteristics one obtains the topological zeta function associated to $\mathcal{I}$.

We attach to an analytic mapping $\boldsymbol{f}=(f_1,\ldots,f_l)$ a Newton polyhedron $\Gamma\left(\boldsymbol{f}\right)$ and a new notion of non-degeneracy with respect to $\Gamma\left(\boldsymbol{f}\right)$. The novelty of this notion resides in the fact that it depends on *one* Newton polyhedron, and Khovanskii's non-degeneracy notion depends on the Newton polyhedra of $f_1,\ldots,f_l$ (see [18], [26]). By constructing a log-principalization, we give an explicit list for the possible poles of $Z_{\Phi}(s,\boldsymbol{f})$, $l\geq 1$, in the case in which $\boldsymbol{f}$ is non degenerate with respect to $\Gamma\left(\boldsymbol{f}\right)$ (see Theorem 3.11). This theorem provides a generalization to the case $l\geq 1$ of a well-known result that describes the poles of the local zeta function associated to a non-degenerate polynomial in terms of the corresponding Newton polyhedron [5], [21], [36]. This result was originally established by Varchenko [31] for local zeta functions over $\mathbb{R}$. If $\boldsymbol{f}$ is non-degenerate with respect to $\Gamma\left(\boldsymbol{f}\right)$, then $\lambda\left(\mathcal{I}_{\boldsymbol{f}}\right)$ can be computed from $\Gamma\left(\boldsymbol{f}\right)$ in the classical way (see Corollary 3.12).

By using our notion of non-degeneracy and toroidal geometry we give an explicit formula for $Z(s,\boldsymbol{f})$ and $Z_0(s,\boldsymbol{f})$, $l\geq 1$. This formula generalizes one given by Denef and Hoornaert in the case $l=1$ [7, Theorem 4.2], and one given by the second author for the local zeta function of a monomial mapping [36, Theorem 6.1].

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2. The Igusa local zeta function of a polynomial mapping

2.1. **Log-principalization and poles of the Igusa local zeta function.** We state the two versions of log-principalization of ideals that we will use in this paper. The first is the 'classical' algebraic formulation, see for example [11], [12], [35]. The second is in the context of $p-$adic analytic functions. It follows from the results in [11], see 5.11 in that paper (noticing that 'Property D' there is valid in the $p-$adic analytic setting).

Theorem 2.1 (Hironaka)**.** *Let X_0 be a smooth algebraic variety over a field of characteristic zero, and $\mathcal{I}$ a sheaf of ideals on X_0. There exists a log-principalization of $\mathcal{I}$, that is a sequence*

$$X_0 \overset{\sigma_1}{\longleftarrow} X_1 \overset{\sigma_2}{\longleftarrow} X_2 \ldots \overset{\sigma_i}{\longleftarrow} X_i \longleftarrow \ldots \overset{\sigma_r}{\longleftarrow} X_r = X$$

of blow-ups $\sigma_i : X_{i-1} \longleftarrow X_i$ in smooth centers $C_{i-1} \subset X_{i-1}$ such that
(1) the exceptional divisor E_i of the induced morphism $\sigma^i = \sigma_1 \circ \ldots \circ \sigma_i : X_i \longrightarrow X_0$ has only simple normal crossings and C_i has simple normal crossings with E_i, and
(2) the total transform $(\sigma^r)^*(\mathcal{I})$ is the ideal of a simple normal crossings divisor $E^{\#}$. If the subscheme determined by $\mathcal{I}$ has no components of codimension one, then $E^{\#}$ is a natural combination of the irreducible components of the divisor E_r.

Remark 2.2. We use notations like $(\sigma^r)^*(\mathcal{I})$ as in [35]. However, other authors use the notation $\mathcal{I}\mathcal{O}_X$ for the same object, for example in [11]. As many other authors we use the term 'log-principalization'. The terms 'principalization' and 'monomialization' are also used for the same purpose by other authors.

Theorem 2.3 ([11])**.** *Let K be a $p-$adic field and U an open submanifold of K^n. Let $f_1, \ldots, f_l$ be $K-$analytic functions on U such that the ideal $\mathcal{I}_{\boldsymbol{f}} = (f_1, \ldots, f_l)$ is not trivial. Then there exists a log-principalization $\sigma : X_K \to U$ of $\mathcal{I}_{\boldsymbol{f}}$, that is,*
(1) X_K is an $n-$dimensional $K-$analytic manifold, σ is a proper $K-$analytic map which is a composition of a finite number of blow-ups in closed submanifolds, and which is an isomorphism outside of the common zero set Z_K of $f_1, \ldots, f_l$;
(2) $\sigma^{-1}(Z_K) = \cup_{i \in T} E_i$, where the E_i are closed submanifolds of X_K of codimension one, each equipped with a pair of positive integers (N_i, v_i) satisfying the following. At every point b of X_K there exist local coordinates $(y_1, \ldots, y_n)$ on X_K around b such that, if $E_1, \ldots, E_p$ are the E_i containing b, we have on some neighborhood of b that E_i is given by $y_i = 0$ for $i = 1, \ldots, p$,

$$\sigma^*(\mathcal{I}_{\boldsymbol{f}}) \text{ is generated by } \varepsilon(y) \prod_{i=1}^{p} y_i^{N_i},$$

and

$$\sigma^*(dx_1 \wedge \ldots \wedge dx_n) = \eta(y) \left(\prod_{i=1}^{p} y_i^{v_i - 1} \right) dy_1 \wedge \ldots \wedge dy_n,$$

where $\varepsilon(y)$, $\eta(y)$ are units in the local ring of X_K at b.

The (N_i, v_i), $i \in T$, are called *the numerical data of σ.*

Let K be a $p-$adic field. Let $f_1, \ldots, f_l$ be polynomials over K or $K-$analytic functions on $U \subset K^n$. We set $\mathcal{I}_{\boldsymbol{f}}$ to be the $K-$analytic ideal generated by the f_i; we suppose it is not trivial. Let $\Phi : K^n \to \mathbb{C}$ or $U \to \mathbb{C}$ be a Schwartz-Bruhat function, that is, a locally constant function with compact support. We associate to $\boldsymbol{f} = (f_1, \ldots, f_l)$ and Φ the Igusa zeta function $Z_\Phi(s, \boldsymbol{f})$ as in the introduction. The following theorem yields a new proof of its meromorphic continuation, but especially it gives a list of its possible poles in terms of the numerical data of a log-principalization.

Theorem 2.4. *The local zeta function $Z_\Phi(s, \boldsymbol{f})$ admits a meromorphic continuation to the complex plane as a rational function of q^{-s}. Furthermore, the poles have the form*

$$s = -\frac{v_i}{N_i} - \frac{2\pi\sqrt{-1}}{\log q}\frac{k}{N_i}, \quad k \in \mathbb{Z},$$

where the (N_i, v_i) are the numerical data of a log-principalization $\sigma : X_K \longrightarrow U$ of the ideal $\mathcal{I}_{\boldsymbol{f}} = (f_1, \ldots, f_l)$.

Proof. We pick a log-principalization σ of $\mathcal{I}_{\boldsymbol{f}}$ as in Theorem 2.3 and we use all notations that were introduced there.

At every point $b \in X_K$ we can take a chart (V, ϕ_V) with coordinates $(y_1, \ldots, y_n)$, which may be schrinked later when necessary. Let $g(y)$ be a generator of $\sigma^*(\mathcal{I}_{\boldsymbol{f}}) = \sigma^*(f_1, \ldots, f_l)$ in V. Then

$$g(y) = \varepsilon(y) \prod_{i=1}^{p} y_i^{N_i},$$

$$\sigma^*(dx_1 \wedge \ldots \wedge dx_n) = \eta(y) \left(\prod_{i=1}^{p} y_i^{v_i - 1} \right) dy_1 \wedge \ldots \wedge dy_n,$$

where $\varepsilon(y)$ and $\eta(y)$ are units of the local ring of X_K at b. Furthermore, since $\sigma^*(\mathcal{I}_{\boldsymbol{f}})$ is locally generated by $g(y)$ we have

$$f_i^*(y) = g(y)\widetilde{f}_i(y),$$

for $i = 1, \ldots, l$, $y \in V$, where each $\widetilde{f}_i(y)$ is an analytic function on V. And, since $g(y) \in \sigma^*(\mathcal{I}_{\boldsymbol{f}})$, we also have $g(y) = \sum_{i=1}^{l} a_i(y) f_i^*(y)$, with $a_i(y)$ an analytic function on V for each i; therefore

$$1 = \sum_{i=1}^{l} a_i(y)\widetilde{f}_i(y), \text{ for } y \in V.$$

Then there exists at least one index i_0 such that $\widetilde{f}_{i_0}(b) \neq 0$, hence we may assume that $\widetilde{f}_{i_0}(y) \neq 0$ on V and that

$$\|(f_1^*(y), \ldots, f_l^*(y))\|_K^s = \left\| \left(\left(\widetilde{f}_i(y)\right)_{i \notin H}, \left(\widetilde{f}_i(b)\right)_{i \in H} \right) \right\|_K^s |g(y)|_K^s,$$

for $y \in V$. Here $H \subseteq \{1, \ldots, n\}$ such that $\widetilde{f}_i(b) \neq 0 \Leftrightarrow i \in H$. We may further suppose that

$$\left\| \left(\left(\widetilde{f}_i(y)\right)_{i \notin H}, \left(\widetilde{f}_i(b)\right)_{i \in H} \right) \right\|_K^s = \left\| \left(\widetilde{f}_i(b)\right)_{i \in H} \right\|_K^s$$

on V. Since σ is proper, $\sigma^{-1}(\operatorname{supp}(\Phi))$ is compact open in X_K, hence we can express it as a finite disjoint union of compact open sets B_α such that each B_α is contained in some V above. Since Φ is locally constant we may assume (after subdividing B_α) that $(\Phi\circ\sigma)\mid_{B_\alpha}=(\Phi\circ\sigma)(b)$, $|\varepsilon|_K\mid_{B_\alpha}=|\varepsilon(b)|_K$, $|\eta|_K\mid_{B_\alpha}=|\eta(b)|_K$, and $\phi_V(B_\alpha)=c+\pi^{e_0}R_K^n$.

Denote by $D_K=(\operatorname{div}(\sigma^*(\mathcal{I}_{\boldsymbol{f}})))_K$. Since $\sigma: X_K\backslash D_K\longrightarrow U\backslash\sigma(D_K)$ is bianalytic, and D_K has measure zero, we have

$$Z_\Phi(s,\boldsymbol{f})=\int\limits_{U\backslash\sigma(D_K)}\Phi(x)\left\|\boldsymbol{f}(x)\right\|_K^s\mid dx\mid$$

$$=\sum_\alpha(\Phi\circ\sigma)(b)\left|\varepsilon(b)\right|_K^s\left|\eta(b)\right|_K\left\|\left(\widetilde{f}_i(b)\right)_{i\in H}\right\|_K^s\int\limits_{c+\pi^{e_0}R_K^n}\prod_{1\leq i\leq p}\left|y_i\right|^{N_is+v_i-1}\left|dy\right|.$$

The conclusion is now obtained by computing the integral in the previous expression like in the case $l=1$ (see [16, Lemma 8.2.1]). ∎

Example 2.5. Let K be a $p-$adic field, and let $f_1(x,y)=y^a-x^b$, $f_2(x,y)=x^a-y^b$, with $a<b$, and for $j=3,\ldots,M$, $M\geq 3$, $f_j(x,y)=x^{n_j}y^{m_j}h_j(x,y)$, with n_j, $m_j\geq a$, and $h_j(x,y)\in K[x,y]$. Set $\boldsymbol{f}=(f_1,f_2,f_3,\ldots,f_M)$, and $I_{\boldsymbol{f}}=(f_1,f_2,f_3,\ldots,f_M)$. Let Φ be a Schwartz-Bruhat function whose support is contained in a sufficiently small neighborhood of the origin. A log-principalization of the ideal $I_{\boldsymbol{f}}$ (over a neighborhood of the origin) is obtained by blowing-up the origin of K^2. There is only one exceptional curve $E=\mathbb{P}^1(K)$ whose numerical datum is $(a,2)$, and therefore the possible poles of $Z_\Phi(s,\boldsymbol{f})$ have real part $\frac{-2}{a}$. In [37] an algorithm for computing a list of candidates for the poles of $Z_\Phi(s,\boldsymbol{f})$ in terms of the numerical data of an embedded resolution of the divisor $\cup_{j=1}^M f_j^{-1}(0)$ was given. Since the $f_j(x,y)$ are arbitrary polynomials for $3\leq j\leq M$, the mentioned algorithm gives in general a very long list of possible poles.

2.2. **The largest real part of the poles of the Igusa zeta function.** Let U be a compact open subset of K^n and let $\boldsymbol{f}=(f_1,\ldots,f_l):U\longrightarrow K^l$ be an analytic mapping. Recall that $Z_U(s,\boldsymbol{f})=\int_U\left\|\boldsymbol{f}(x)\right\|_K^s|dx|$. The following lemma is known by the experts, however we did not find a suitable reference for it; for the sake of completeness we include its proof here.

Lemma 2.6. *(1) $Z_U(s,\boldsymbol{f})$ has no pole in s, i.e. $Z_U(s,\boldsymbol{f})$ is a Laurent polynomial in q^{-s} if and only if there is no $x\in U$ such that $f_1(x)=\ldots=f_l(x)=0$.*
(2) If $0\in U$ and $\boldsymbol{f}(0)=0$, i.e. $f_1(0)=\ldots=f_l(0)=0$, then $Z_U(s,\boldsymbol{f})$ has at least one pole in s.

Proof. (1) We first note that rationality of $Z_U(s,\boldsymbol{f})$ implies the equivalence of the conditions "$Z_U(s,\boldsymbol{f})$ has no pole in s" and "$Z_U(s,\boldsymbol{f})$ is a Laurent polynomial in q^{-s}."
($\Leftarrow$) Since $\boldsymbol{f}:U\longrightarrow K^l$ is continuous, also $\left\|\boldsymbol{f}\right\|_K:U\longrightarrow q^{\mathbb{Z}}\cup\{0\}$ is continuous. If 0 does not belong to the image of $\left\|\boldsymbol{f}\right\|_K$, then there are only finitely many values in the image because U is compact. So $\int_U\left\|\boldsymbol{f}(x)\right\|_K^s|dx|$ is a Laurent polynomial in q^{-s}.
($\Rightarrow$) If $x_0\in U$ with $f_1(x_0)=\ldots=f_l(x_0)=0$, by using the continuity of $\left\|\boldsymbol{f}\right\|_K$, there exist infinitely many i such that there exists $x_i\in U$ with $\left\|\boldsymbol{f}(x_i)\right\|_K=q^{-i}$.

Since U is open we have for all those i that the Haar measure of the set

$$\left\{x \in U \mid \|\boldsymbol{f}(x)\|_K = q^{-i}\right\}$$

is positive. Therefore

$$Z_U(s, \boldsymbol{f}) = \sum_j vol\left(\left\{x \in U \mid \|\boldsymbol{f}(x)\|_K = q^{-j}\right\}\right)\ q^{-sj}$$

is not a Laurent polynomial in q^{-s}.

(2) The second part follows directly from the first one. ∎

Theorem 2.7. *Let $\boldsymbol{f} = (f_1, \ldots, f_l) : U \longrightarrow K^l$ be an analytic mapping defined on a compact open neighborhood of the origin U such that $\boldsymbol{f}(0) = 0$. We take a log-principalization $\sigma : X_K \to U$ as in Theorem 2.3 with numerical data (N_i, v_i), $i \in T$. Let $\lambda := \lambda(\mathcal{I}_{\boldsymbol{f}}) = \min_i \frac{v_i}{N_i}$. Then $-\lambda(\mathcal{I}_{\boldsymbol{f}})$ is the real part of a pole of $Z_U(s, \boldsymbol{f})$. In particular, $\lambda(\mathcal{I}_{\boldsymbol{f}})$ depends only on $\mathcal{I}_{\boldsymbol{f}}$.*

Proof. The proof will be achieved by establishing that q^{λ} is the radius of convergence $\mathtt{R}$ of $Z_U(s, \boldsymbol{f})$ considered as a function in q^{-s}. Certainly $\mathtt{R} \geq q^{\lambda}$, since (by Theorem 2.4) the candidate poles closest to the origin have modulus q^{λ}. We shall show that $\mathtt{R} \leq q^{\lambda}$ by proving a lower bound for the coefficients of $Z_U(q^{-s}, \boldsymbol{f})$, considered as power series in q^{-s}:

$$Z_U\left(q^{-s}, \boldsymbol{f}\right) = \sum_j vol\left(\left\{x \in U \mid \|\boldsymbol{f}(x)\|_K = q^{-j}\right\}\right)\ q^{-sj}.$$

Take a *generic* point b on a component E_r with $\frac{v_r}{N_r} = \lambda$, and a small enough chart B $(\subset X_K)$ around b with coordinates $(y_1, \ldots, y_n)$ such that

$$\sigma^*(\mathcal{I}_{\boldsymbol{f}}) \text{ is generated by } \varepsilon(y)\, y_1^{N_r},$$

and

$$\sigma^*(dx_1 \wedge \ldots \wedge dx_n) = \eta(y)\, y_1^{v_r - 1} dy_1 \wedge \ldots \wedge dy_n,$$

on B, where $|\varepsilon|_K$ and $|\eta|_K$ are constant (and nonzero) on B. After an eventual $K-$analytic coordinate change, we may assume furthermore that $B = R_K^n$.

Claim. *For j big enough and divisible by N_r we have*

$$vol\left(\left\{x \in U \mid \|\boldsymbol{f}(x)\|_K = q^{-j}\right\}\right) \geq C q^{-j\lambda},$$

where C is a positive constant.

By the above claim we have

$$\limsup_{i \to \infty} \left[vol\left(\left\{x \in U \mid \|\boldsymbol{f}(x)\|_K = q^{-i}\right\}\right)\right]^{1/i} \geq q^{-\lambda}$$

and hence

$$\mathtt{R} = \frac{1}{\limsup_{i \to \infty} \left[vol\left(\left\{x \in U \mid \|\boldsymbol{f}(x)\|_K = q^{-i}\right\}\right)\right]^{1/i}} \leq q^{\lambda}.$$

Therefore, since $Z_U(q^{-s}, \boldsymbol{f})$ is a rational function of q^{-s}, we conclude that uq^{λ} is a pole of $Z_U(q^{-s}, \boldsymbol{f})$, for some complex N_r-th root of the unity u.

Proof of the claim. By the $p-$adic change of variables formula [16, Proposition 7.4.1] we have $(B \subset \sigma^{-1}(U))$:

$$vol\left(\left\{x \in U \mid \|\boldsymbol{f}(x)\|_K = q^{-j}\right\}\right) \geq$$

$$vol\left(\left\{y\in B\mid \left\Vert \boldsymbol{f}\circ\sigma\left(y\right)\right\Vert_K=q^{-j}\right\}\right)\cdot\left\vert\left(Jac\ \sigma\right)\left(y\right)\right\vert_K, \tag{2.1}$$

where $Jac\ \sigma$ is the Jacobian determinant of σ. With the same reasoning as in the proof of Theorem 2.4 we have that $\left\Vert \boldsymbol{f}\circ\sigma\left(y\right)\right\Vert_K=C_1\left\vert\varepsilon\right\vert_K\left\vert y_1\right\vert_K^{N_r}$ on B, where C_1 is a positive constant. So on B we have $\left\Vert \boldsymbol{f}\circ\sigma\left(y\right)\right\Vert_K=q^{-j}$ if and only if $\left\vert y_1\right\vert_K=C_2q^{-j/N_r}$, where C_2 is a positive constant. Hence

$$vol\left(\left\{y\in B\mid \left\Vert \boldsymbol{f}\circ\sigma\left(y\right)\right\Vert_K=q^{-j}\right\}\right)=\left(1-q^{-1}\right)C_2q^{-j/N_r}. \tag{2.2}$$

Note that on this subset of B we have

$$\left\vert\left(Jac\ \sigma\right)\left(y\right)\right\vert_K=\left\vert\eta\right\vert_K\left\vert y_1\right\vert_K^{v_r-1}=\left\vert\eta\right\vert_K C_2^{v_r-1}q^{-j(v_r-1)/N_r}. \tag{2.3}$$

Combining (2.1), (2.2) and (2.3) yields

$$vol\left(\left\{x\in U\mid \left\Vert \boldsymbol{f}\left(x\right)\right\Vert_K=q^{-j}\right\}\right)\geq Cq^{-\lambda j},$$

for some positive constant C. ∎

Remark 2.8. (1) In [15] Igusa showed in the case $l=1$ that $-\lambda\left(\mathcal{I}_{\boldsymbol{f}}\right)$ is a pole of $Z_U\left(s,\boldsymbol{f}\right)$ for a suitable compact open set U containing the origin. The argument uses Langlands' description of residues in terms of principal value integrals [20]. Furthermore, this argument is valid for archimedean and non-archimedean local zeta functions (see also [2, Théorème 5, part 3a, page 186], [31]).
(2) We note that $\lambda(\mathcal{I}_{\boldsymbol{f}})\geq lct(\mathcal{I}_{\boldsymbol{f}})$, where $lct(\mathcal{I}_{\boldsymbol{f}})$ is the 'log-canonical threshold' of $\mathcal{I}_{\boldsymbol{f}}$. This well-known important invariant (see e.g. [19], [25]) is defined analogously as $\lambda(\mathcal{I}_{\boldsymbol{f}})$ but in a geometric setting, i.e. working over an algebraic closure of K. In order to obtain a log-principalization in this context maybe more exceptional components are needed, and then the inequality above could be strict.

2.2.1. *Number of solutions of polynomial congruences.* Suppose that $f_i(x)$, $i=1,...,l$, are polynomials with coefficients in R_K. Let $N_j(\boldsymbol{f})$ be the number of solutions of $f_i(x)\equiv 0 \mod P_K^j$, $i=1,...,l$, in $\left(R_K/P_K^j\right)^n$, and let $P(t,\boldsymbol{f})$ be the series $\sum_{j=0}^{\infty}N_j(\boldsymbol{f})(q^{-n}t)^j$. The Poincaré series $P(t,\boldsymbol{f})$ is related to $Z(s,\boldsymbol{f})$ by the formula $P(t,\boldsymbol{f})=\frac{1-tZ(s,\boldsymbol{f})}{1-t}$, $t=q^{-s}$, (cf. [24, Theorem 2]). In the proof of the previous theorem was established that q^{λ} is the radius of convergence $\mathtt{R}$ of $Z\left(s,\boldsymbol{f}\right)$ considered as a function in q^{-s}. By using this fact, and the above-mentioned relation between $P(t,\boldsymbol{f})$ and $Z(s,\boldsymbol{f})$, we obtain the following corollary.

Corollary 2.9. *With the above notation,*

$$\limsup_{j\to\infty}\left[N_j(\boldsymbol{f})q^{-nj}\right]^{\frac{1}{j}}=q^{-\lambda(\mathcal{I}_{\boldsymbol{f}})},$$

where $\lambda\left(\mathcal{I}_{\boldsymbol{f}}\right)=\min\left\{\frac{v_i}{N_i}\right\}$, *where* (N_i,v_i) *runs through the numerical data of a log-principalization* $\sigma:X_K\longrightarrow R_K^n$ *of the ideal* $\mathcal{I}_{\boldsymbol{f}}=(f_1,\ldots,f_l)$.

Let d be the maximal order of the poles of $P(t,\boldsymbol{f})$ with modulus $q^{\lambda(\mathcal{I}_{\boldsymbol{f}})}$. As a consequence of the above corollary and of the rationality of $P(t,\boldsymbol{f})$ we have that $N_j(\boldsymbol{f})\leq Cj^{d-1}q^{(n-\lambda(\mathcal{I}_{\boldsymbol{f}}))j}$ for j big enough, where C is a positive constant. And by Remark 2.8 (2), we have then that $N_j(\boldsymbol{f})\leq Cj^{d-1}q^{(n-lct(\mathcal{I}_{\boldsymbol{f}}))j}$ for j big enough.

2.3. **Denef's explicit formula.** For polynomials $f_1, \ldots, f_l$ over a number field F, we can consider local zeta functions $Z_W(s, \boldsymbol{f}, K)$ for all (non-archimedean) completions K of F. When $l = 1$, Denef presented in [6, Theorem 3.1] an explicit formula, which is valid simultaneously for almost all these zeta functions. His arguments extend to the several polynomials case, by replacing resolution by log-principalization (as in Theorem 2.1).

Theorem 2.10. *Let F be a number field and $f_i(x) \in F[x_1, \ldots, x_n]$ for $i = 1, \ldots, l$. Let $\sigma : X \to \mathbb{A}^n$ be a log-principalization of $I_{\boldsymbol{f}} = (f_1, \ldots, f_l)$ over F as in Theorem 2.1. Denote div$(\sigma^*(\mathcal{I}_{\boldsymbol{f}})) = \sum_{i\in T} N_i E_i$, and div$(\sigma^*(dx_1 \wedge \ldots \wedge dx_n)) = \sum_{i\in T}(v_i - 1) E_i$, where E_i, $i \in T$, are the irreducible components of the simple normal crossings divisor given by the principal ideal $\sigma^*(\mathcal{I}_{\boldsymbol{f}})$. For every maximal ideal P of the ring of integers of F, we consider the completion K of F with respect to P. Denote the valuation ring and the residue field of K by R and $\overline{K} = \mathbb{F}_q$ respectively. Then for almost all completions K (i.e. for all except a finite number) we have*

$$Z_W(s, \boldsymbol{f}, K) = q^{-n} \sum_{I\subseteq T} c_I \prod_{i\in I} \frac{(q-1)\, q^{-N_i s - v_i}}{1 - q^{-N_i s - v_i}},$$

where $W \subset R^n$ is a union of cosets mod $(P)^n$, and

$$c_I = card\left\{a \in \overline{X}\left(\overline{K}\right) \mid a \in \overline{E_i}\left(\overline{K}\right) \Leftrightarrow i \in I;\ and\ \overline{\sigma}(a) \in \overline{W}\right\}.$$

Here $\bar{\ }$ denotes the reduction mod P, for which we refer to [6, Sect. 2].

Example 2.11. Take $f_1, f_2, f_3, \ldots, f_M$ as in Example 2.5 as being defined over a number field F. Then the formula of Theorem 2.10 for $W = (P)^2$ yields

$$Z_0(s, \boldsymbol{f}, K) = q^{-2}(q+1)\frac{(q-1)\, q^{-as-2}}{1 - q^{-as-2}} = \frac{\left(1 - q^{-2}\right) q^{-as-2}}{1 - q^{-as-2}}.$$

Example 2.12. Let $K = \mathbb{Q}_p$, $f_1(x, y) = x$, $f_2(x, y) = x + p_0 y$, where p_0 is a fixed prime number, and let $\boldsymbol{f} = (f_1, f_2)$. A direct calculation shows that

$$Z(s, \boldsymbol{f}, K) = \begin{cases} \frac{1-p^{-2}}{1-p^{-2-s}}, & p \neq p_0, \\ \\ \frac{\left(1-p^{-1}\right)\left(1+p^{-1-s}\right)}{1-p^{-2-s}}, & p = p_0. \end{cases}$$

A log-principalization for the ideal $\mathcal{I}_{\boldsymbol{f}}$ is attained by blowing-up the origin. One easily verifies that the expression for $p \neq p_0$ is the one given by Theorem 2.10.

As a consequence of Theorem 2.4 (or [4], [24]) $Z_W(s, \boldsymbol{f})$ can be written as

$$Z_W(s, \boldsymbol{f}) = \frac{P(T)}{Q(T)},$$

where $P(T)$ and $Q(T)$ are polynomials in $T = q^{-s}$ with rational coefficients. We define deg $Z_W(s, \boldsymbol{f}) =$ deg $P(T) -$ deg $Q(T)$, where deg means 'degree'.

Corollary 2.13. *Let $f_i(x) \in F[x_1, \ldots, x_n]$ for $i = 1, \ldots, l$. For almost all completions K of F we have* deg $Z(s, \boldsymbol{f}, K) \le 0$ *and* deg $Z_0(s, \boldsymbol{f}, K) = 0$. *Moreover if all f_i are homogeneous of degree d, then* $\deg Z(s, \boldsymbol{f}, K) = -d$.

The proof follows from the explicit formula (Theorem 2.10) by analogous arguments as in [6] (or [16]) where the case $l = 1$ is treated. We should mention that by using model-theoretic arguments Denef already showed the above result (see

[6, Theorem 5.2, and Example 5.4]). So in this paper we give a geometric proof of this fact.

Note that for the case $p = p_0$ in Example 2.12 it is not true that $\deg Z(s, \boldsymbol{f}, \mathbb{Q}_p) = -1$, though f_1, f_2 are homogeneous of degree 1.

Example 2.14. Let $\boldsymbol{f} = (f_1, f_2) = \left(x^3 - xy, y\right)$. One easily constructs a log-principalization of the ideal $\mathcal{I}_{\boldsymbol{f}} = \left(x^3 - xy, y\right)$ as a composition of three blow-ups. The numerical data of the three exceptional components in $\sigma^{-1}\left(\operatorname{supp} \mathcal{I}_{\boldsymbol{f}}\right) = \sigma^{-1}(0)$ are $(1,2)$, $(2,3)$, $(3,4)$ respectively. So Theorem 2.4 yields -2, $-3/2$, $-4/3$ as possible (real parts of) candidate poles of $Z(s, \boldsymbol{f})$. However, in the formula of Theorem 2.10 the first two candidate poles cancel:

$$\begin{aligned} Z(s,\boldsymbol{f}) &= q^{-2}\Big\{\left(q^2-1\right) + q\frac{(q-1)\,q^{-2-s}}{1-q^{-2-s}} + (q-1)\,\frac{(q-1)\,q^{-3-2s}}{1-q^{-3-2s}} + q\frac{(q-1)\,q^{-4-3s}}{1-q^{-4-3s}} \\ &\quad + \frac{(q-1)^2\,q^{-5-3s}}{\left(1-q^{-2-s}\right)\left(1-q^{-3-2s}\right)} + \frac{(q-1)^2\,q^{-7-5s}}{\left(1-q^{-3-2s}\right)\left(1-q^{-4-3s}\right)}\Big\} \\ &= q^{-2}\frac{q-1}{1-q^{-4-3s}}\left(q+1+q^{-1-s}+q^{-2-2s}\right). \end{aligned}$$

We shall present an alternative formula to compute this example in Section 4, where only one candidate pole will appear.

Example 2.15. Let $\boldsymbol{f} = (f_1, f_2) = \left(y^2 - x^3, y^2 - z^2\right)$. We shall compute $Z_0(s, \boldsymbol{f})$ by means of a log-principalization of $\mathcal{I}_{\boldsymbol{f}} = \left(y^2 - x^3, y^2 - z^2\right)$. Note that the support of $\mathcal{I}_{\boldsymbol{f}}$ has two 1−dimensional components C and C' with a singularity at the origin of K^3.

We first blow up the origin yielding the exceptional surface $E_1 \left(\cong \mathbb{P}^2\right)$ with $(N_1, v_1) = (2,3)$. The strict transform of C and C' and E_1 have one common point. Next we blow up this point obtaining the new exceptional surface $E_2 \left(\cong \mathbb{P}^2\right)$ with $(N_2, v_2) = (3,5)$. At this stage (the strict transforms of) C and C' are disjoint and both meet E_2 in one point of the intersection of E_2 with (the strict transform of) E_1. Now we blow up the curve $E_1 \cap E_2$; the new exceptional component E_3 is a ruled surface over that curve and $(N_3, v_3) = (6,8)$. We have that $E_3 \cap E_1$ and $E_3 \cap E_2$ are disjoint sections of E_3, and C and C' intersect E_3 transversely outside $E_3 \cap E_1$ and $E_3 \cap E_2$. Finally we blow up C and C', yielding the last two exceptional surfaces E_4 and E_4' with numerical data $(1,2)$. The formula of Theorem 2.10 yields

$$\begin{aligned} Z_0(s,\mathbf{f}) &= q^{-3}\Bigg((q^2+q)\frac{(q-1)q^{-3-2s}}{1-q^{-3-2s}} + q^2\frac{(q-1)q^{-5-3s}}{1-q^{-5-3s}} \\ &\quad + (q^2-3)\frac{(q-1)q^{-8-6s}}{1-q^{-8-6s}} + (q+1)\frac{(q-1)^2q^{-11-8s}}{(1-q^{-3-2s})(1-q^{-8-6s})} \\ &\quad + (q+1)\frac{(q-1)^2q^{-13-9s}}{(1-q^{-5-3s})(1-q^{-8-6s})} + 2(q+1)\frac{(q-1)^2q^{-10-7s}}{(1-q^{-2-s})(1-q^{-8-6s})}\Bigg) \\ &= q^{-3}(q-1)\frac{N\left(q^{-s}\right)}{(1-q^{-2-s})(1-q^{-8-6s})}, \end{aligned}$$

where

$$\begin{aligned} N\left(q^{-s}\right) &= (q^2-q-1)q^{-10-7s} + (q^2+q-1)q^{-8-6s} - (q+1)q^{-7-5s} \\ &\quad + q^{-4-4s} - q^{-4-3s} + (q+1)q^{-2-2s}. \end{aligned}$$

Note that the candidate poles $-3/2$ and $-5/3$ cancel.

2.4. **Motivic and topological zeta functions.** The analogue of the original explicit formula of Denef plays an important role in the study of the motivic zeta function associated to one regular function [8]. One can associate more generally a motivic zeta function to any sheaf of ideals on a smooth variety, and obtain a similar formula for it in terms of a log-principalization using the argument of [8]. We just formulate the more general definition and formula, referring to e.g. [9], [32] for the notion of jets and Grothendieck ring.

Definition 2.16. Let Y be a smooth algebraic variety of dimension n over over a field F of characteristic zero, and $\mathcal{I}$ a sheaf of ideals on Y. Let W be a subvariety of Y. Denote for $i \in \mathbb{N}$ by $\mathfrak{X}_{i,W}$ the variety of $i-$jets γ on Y with origin in W for which $ord_t(\gamma^*\mathcal{I}) = i$. The *motivic zeta function* associated to $\mathcal{I}$ (and W) is the formal power series

$$Z_W(\mathcal{I}, T) = \sum_{i \geq 0} [\mathfrak{X}_{i,W}] \left(\mathbb{L}^{-n}T\right)^i,$$

where $[\cdot]$ denotes the class of a variety in the Grothendieck ring of algebraic varieties over F, and $\mathbb{L} = \left[\mathbb{A}^1\right]$.

Theorem 2.17. *Let $\sigma : X \to Y$ be a log-principalization of $\mathcal{I}$. With the analogous notation E_i, N_i, v_i, $(i \in T)$ as before, and also $E_I^\circ := (\cap_{i\in I} E_i) \setminus (\cup_{k \notin I} E_k)$ for $I \subset T$, we have*

$$Z_W(\mathcal{I}, T) = \sum_{I \subset T} \left[E_I^\circ \cap \sigma^{-1}W\right] \prod_{i \in I} \frac{(\mathbb{L}-1)\, T^{N_i}}{\mathbb{L}^{v_i} - T^{N_i}}.$$

In particular $Z_W(\mathcal{I}, T)$ is rational in T.

Specializing to topological Euler characteristics, denoted by $\chi(\cdot)$, as in [8, (2.3)] or [32, (6.6)] we obtain the expression

$$Z_{top,W}(\mathcal{I}, s) := \sum_{I \subset T} \chi\left(E_I^\circ \cap \sigma^{-1}W\right) \prod_{i \in I} \frac{1}{v_i + N_i s} \in \mathbb{Q}(s),$$

which is then independent of the chosen log-principalization. (When the base field is not the complex numbers, we consider $\chi(\cdot)$ in étale $\overline{\mathbb{Q}}$- cohomology as in [8].) It can be taken as a definition for the *topological zeta function* associated to $\mathcal{I}$ (and W), generalizing the original one of Denef and Loeser associated to one polynomial [10].

3. Newton polyhedra and non-degeneracy conditions

3.1. **Newton polyhedra.** We set $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x \geqslant 0\}$.

Let G be a nonempty subset of $\mathbb{N}^n$. The *Newton polyhedron* $\Gamma = \Gamma(G)$ associated to G is the convex hull in $\mathbb{R}_+^n$ of the set $\cup_{m \in G}\left(m + \mathbb{R}_+^n\right)$. For instance classically one associates a *Newton polyhedron (at the origin) to* $g(x) = \sum_m c_m x^m$ $(x = (x_1, \ldots, x_n)$, $g(0) = 0)$, being a nonconstant polynomial function over K or $K-$analytic function in a neighborhood of the origin, where $G = \text{supp}(g) := \{m \in \mathbb{N}^n \mid c_m \neq 0\}$. Further we will associate more generally a Newton polyhedron to an analytic mapping.

We fix a Newton polyhedron Γ as above. We first collect some notions and results about Newton polyhedra that will be used in the next sections. Let $\langle \cdot, \cdot \rangle$

denote the usual inner product of $\mathbb{R}^n$, and identify the dual space of $\mathbb{R}^n$ with $\mathbb{R}^n$ itself by means of it.

For $a \in \mathbb{R}_+^n$, we define

$$d(a,\Gamma) = d(a) = \min_{x\in\Gamma} \langle a, x\rangle\,,$$

and *the first meet locus* $F(a)$ of a as

$$F(a) := \{x \in \Gamma \mid \langle a, x\rangle = d(a)\}.$$

The first meet locus is a face of Γ. Moreover, if $a \neq 0$, $F(a)$ is a proper face of Γ.

We define an equivalence relation in $\mathbb{R}_+^n$ by taking $a \sim a' \Leftrightarrow F(a) = F(a')$. The equivalence classes of $\sim$ are sets of the form

$$\Delta_\tau = \{a \in \mathbb{R}_+^n \mid F(a) = \tau\},$$

where τ is a face of Γ.

We recall that the cone strictly spanned by the vectors $a_1, \ldots, a_r \in \mathbb{R}_+^n \setminus \{0\}$ is the set $\Delta = \{\lambda_1 a_1 + ... + \lambda_r a_r \mid \lambda_i \in \mathbb{R}_+,\, \lambda_i > 0\}$. If $a_1, \ldots, a_r$ are linearly independent over $\mathbb{R}$, Δ is called a *simplicial cone*. If $a_1, \ldots, a_r \in \mathbb{Z}^n$, we say Δ is a *rational cone*. If $\{a_1, \ldots, a_r\}$ is a subset of a basis of the $\mathbb{Z}$-module $\mathbb{Z}^n$, we call Δ a *simple cone*.

A precise description of the geometry of the equivalence classes modulo $\sim$ is as follows. Each *facet* (i.e. a face of codimension one) γ of Γ has a unique vector $a(\gamma) = (a_{\gamma,1}, \ldots, a_{\gamma,n}) \in \mathbb{N}^n \setminus \{0\}$, whose nonzero coordinates are relatively prime, which is perpendicular to γ. We denote by $\mathfrak{D}(\Gamma)$ the set of such vectors. The equivalence classes are rational cones of the form

$$\Delta_\tau = \{\sum_{i=1}^{r} \lambda_i a(\gamma_i) \mid \lambda_i \in \mathbb{R}_+,\, \lambda_i > 0\},$$

where τ runs through the set of faces of Γ, and γ_i, $i = 1, \ldots, r$ are the facets containing τ. We note that $\Delta_\tau = \{0\}$ if and only if $\tau = \Gamma$. The family $\{\Delta_\tau\}_\tau$, with τ running over the proper faces of Γ, is a partition of $\mathbb{R}_+^n \setminus \{0\}$; we call this partition a *polyhedral subdivision of* $\mathbb{R}_+^n$ *subordinated* to Γ. We call $\left\{\overline{\Delta_\tau}\right\}_\tau$, the family formed by the topological closures of the Δ_τ, a *fan subordinated* to Γ.

Each cone Δ_τ can be partitioned into a finite number of simplicial cones $\Delta_{\tau,i}$. In addition, the subdivision can be chosen such that each $\Delta_{\tau,i}$ is spanned by part of $\mathfrak{D}(\Gamma)$. Thus from the above considerations we have the following partition of $\mathbb{R}_+^n \setminus \{0\}$:

$$\mathbb{R}_+^n \setminus \{0\} = \bigcup_\tau \left(\bigcup_{i=1}^{l_\tau} \Delta_{\tau,i} \right), \tag{3.1}$$

where τ runs over the proper faces of Γ, and each $\Delta_{\tau,i}$ is a simplicial cone contained in Δ_τ. We will say that $\{\Delta_{\tau,i}\}$ is a *simplicial polyhedral subdivision of* $\mathbb{R}_+^n$ *subordinated* to Γ; and that $\left\{\overline{\Delta_{\tau,i}}\right\}$ is a *simplicial fan subordinated* to Γ.

By adding new rays , each simplicial cone can be partitioned further into a finite number of simple cones. In this way we obtain a *simple polyhedral subdivision* of $\mathbb{R}_+^n$ *subordinated* to Γ; and a *simple fan subordinated* to Γ (see e.g. [17]).

3.2. **The Newton polyhedron associated to an analytic mapping.** Let $\boldsymbol{f} = (f_1, \ldots, f_l)$, $\boldsymbol{f}(0) = 0$, be a nonconstant polynomial mapping, or more generally, an analytic mapping defined on a neighborhood $U \subseteq K^n$ of the origin. In this paper we associate to $\boldsymbol{f}$ a Newton polyhedron $\Gamma(\boldsymbol{f}) := \Gamma\left(\cup_{i=1}^{l} \mathrm{supp}\,(f_i)\right)$, and a non-degeneracy condition to $\boldsymbol{f}$ and $\Gamma(\boldsymbol{f})$.

If $f_i(x) = \sum_m c_{m,i} x^m$, and τ is a face of $\Gamma(\boldsymbol{f})$, we set

$$f_{i,\tau}(x) := \sum_{m \in \mathrm{supp}(f_i) \cap \tau} c_{m,i} x^m.$$

Definition 3.1. (1) Let $\boldsymbol{f} = (f_1, \ldots, f_l) : U \longrightarrow K^l$ be a nonconstant analytic mapping satisfying $\boldsymbol{f}(0) = 0$. The mapping $\boldsymbol{f}$ is called *strongly non-degenerate at the origin with respect to* $\Gamma(\boldsymbol{f})$, if for any *compact* face $\tau \subset \Gamma(\boldsymbol{f})$ and any $z \in \left\{z \in (K^\times)^n \mid f_{1,\tau}(z) = \ldots = f_{l,\tau}(z) = 0\right\}$ it verifies that $rank_K \left[\frac{\partial f_{i,\tau}}{\partial x_j}(z)\right] = \min\{l, n\}$.
(2) Let $\boldsymbol{f} = (f_1, \ldots, f_l) : K^n \longrightarrow K^l$ be a nonconstant polynomial mapping satisfying $\boldsymbol{f}(0) = 0$. The mapping $\boldsymbol{f}$ is called *strongly non-degenerate with respect to* $\Gamma(\boldsymbol{f})$, if for any face $\tau \subset \Gamma(\boldsymbol{f})$, including $\Gamma(\boldsymbol{f})$ itself, and any $z \in \left\{z \in (K^\times)^n \mid f_{1,\tau}(z) = \ldots = f_{l,\tau}(z) = 0\right\}$ it verifies that $rank_K \left[\frac{\partial f_{i,\tau}}{\partial x_j}(z)\right] = \min\{l, n\}$.

Remark 3.2. Let $\boldsymbol{f} = (f_1, \ldots, f_l) : U \longrightarrow K^l$ be a nonconstant analytic mapping satisfying $\boldsymbol{f}(0) = 0$.
(1) Let γ be a face of $\Gamma(\boldsymbol{f})$ for which the rank condition in Definition 3.1 is satisfied. If $\mathrm{supp}(f_i) \cap \gamma \neq \varnothing \Leftrightarrow i \in I_\gamma$ for a non-empty subset $I_\gamma \subseteq \{1, \ldots, l\}$ satisfying $\mathrm{card}(I_\gamma) < \min\{l, n\}$, then necessarily

$$\bigcap_{i \in I_\gamma} \left\{z \in \left(K^\times\right)^n \mid f_{i,\gamma}(z) = 0\right\} = \varnothing.$$

(2) If for a given face γ at least one $f_{i,\gamma}$ is a monomial, then the rank condition on γ is satisfied. This is in particular true if γ is a point.

Example 3.3. Let $\boldsymbol{f}(x, y) = \left(x^3 - xy, y\right)$. The mapping $\boldsymbol{f}$ is strongly non-degenerate at the origin with respect to $\Gamma(\boldsymbol{f})$, and also strongly non-degenerate with respect to $\Gamma(\boldsymbol{f})$.

Example 3.4. Let $\boldsymbol{f}(x, y, z) = \left(x^2, y^2, z^2, xy, xz, yz\right)$. Then $\boldsymbol{f}$ is strongly non-degenerate at the origin with respect to $\Gamma(\boldsymbol{f})$, and also strongly non-degenerate with respect to $\Gamma(\boldsymbol{f})$,

3.2.1. *Monomial mappings.* Any monomial mapping is strongly non-degenerate at the origin with respect to its Newton polyhedron. If $\boldsymbol{f}_0$ is a fixed monomial mapping with Newton polyhedron $\Gamma(\boldsymbol{f}_0)$, and $\boldsymbol{f} = \boldsymbol{f}_0 + \boldsymbol{g}$ is a deformation of $\boldsymbol{f}_0$ such that all the monomials in $\boldsymbol{g}$ have exponents in the interior of $\Gamma(\boldsymbol{f}_0)$, then $\boldsymbol{f}$ is strongly non-degenerate at the origin with respect to $\Gamma(\boldsymbol{f}) = \Gamma(\boldsymbol{f}_0)$. This type of mapping was introduced by the second author in [37, Definition 6.1]. Furthermore, the corresponding local zeta function can be computed by using a simple polyhedral subdivision subordinated to $\Gamma(\boldsymbol{f}_0)$ [37, Theorem 6.1].

3.2.2. *Saia's non-degeneracy condition.* In [28] Saia introduced the following notion of non-degeneracy for ideals. Let $I = (f_1, \ldots f_l)$ be a polynomial ideal. I is non-degenerate with respect to $\Gamma(I)$ (where $\Gamma(I) = \Gamma\left(\cup_{i=1}^{l} \mathrm{supp}\,(f_i)\right)$), if for every

compact face τ of $\Gamma(I)$, the system of equations $f_{1,\tau}(z)=0,\dots f_{l,\tau}(z)=0$ does not have a solution in the torus $(K^{\times})^n$. Thus Saia's notion of non-degeneracy is a particular case of our notion of non-degeneracy. Saia's notion of non-degeneracy plays an important role in the study of the integral closure of ideals.

3.2.3. *Khovanskii's non-degeneracy condition.* Now we discuss the relation between our notion of non-degeneracy and Khovanskii's notion of non-degeneracy of an analytic mapping with respect to several Newton polyhedra ([18], see also [26]). Given a positive vector a (i.e. $a\in(\mathbb{N}\setminus\{0\})^n$), and an analytic mapping g, we set $g_a(x):=g_{F(a)}(x)$, where $F(a)$ is the first meet locus of a with respect to $\Gamma(g)$. To make explicit the dependence between $F(a)$ and $\Gamma(g)$ we shall write $F(a,\Gamma(g))$ instead of $F(a)$.

Definition 3.5. A nonconstant analytic mapping $\boldsymbol{f}=(f_1,\dots,f_l):U\longrightarrow K^l$, $\boldsymbol{f}(0)=0$, is non-degenerate with respect to $(\Gamma(f_1),\dots,\Gamma(f_l))$, if for any positive vector a and any $z\in\left\{z\in(K^{\times})^n \mid f_{1,a}(z)=\ldots=f_{l,a}(z)=0\right\}$ it verifies that

$$rank_K\left[\frac{\partial f_{i,a}}{\partial x_j}(z)\right]=\min\{l,n\}.$$

Here $f_{j,a}(z)=f_{j,F(a,\Gamma(f_j))}(z)$ for every j.

The above definition is equivalent to the non-degeneracy notion given by Oka in [26], that is in turn a reformulation of the notion of non-degeneracy introduced by Khovanskii in [18].

Remark 3.6. Let $\boldsymbol{f}=(f_1,\dots,f_l):U\longrightarrow K^l$ be a nonconstant analytic mapping satisfying $\boldsymbol{f}(0)=0$. Then $\Gamma(\boldsymbol{f})$ is the convex hull in $(\mathbb{R}_+)^n$ of $\cup_{j=1}^{l}\Gamma(f_j)$. This assertion follows from the fact that for any subsets A, $B\subseteq(\mathbb{R}_+)^n$, $\overline{A\cup B}=\overline{\overline{A}\cup\overline{B}}$, where the bar denotes the convex hull in $(\mathbb{R}_+)^n$.

The following is the relation between Khovanskii's non-degeneracy notion and the one introduced here.

Proposition 3.7. *Let $\boldsymbol{f}=(f_1,\dots,f_l):U\longrightarrow K^l$ be an analytic mapping strongly non-degenerate at the origin with respect to $\Gamma(\boldsymbol{f})$. Then $\boldsymbol{f}$ is non-degenerate with respect to*

$$(\Gamma(f_1),\dots,\Gamma(f_l)).$$

Proof. Let $a\in(\mathbb{N}\setminus\{0\})^n$ be a fixed positive vector. We set $\Gamma=\Gamma(\boldsymbol{f})$, $\Gamma_j=\Gamma(f_j)$, $j=1,\dots,l$. Since $\Gamma_j\subseteq\Gamma$ by the above remark,

$$d(a,\Gamma)=\min_{x\in\Gamma}\langle a,x\rangle\le d(a,\Gamma_j)=\min_{x\in\Gamma_j}\langle a,x\rangle,$$

for $j=1,\dots,l$. We define $I\subseteq\{1,\dots,l\}$ by the condition

$$j\in I\Leftrightarrow d(a,\Gamma)=d(a,\Gamma_j).$$

Note that $I\neq\varnothing$. Then, if $\tau:=F(a,\Gamma)$,

$$F(a,\Gamma_j)\subseteq\tau,\text{ for } j\in I,$$

and

$$f_{j,\tau}(x)=\begin{cases} f_{j,a}(x), & j\in I,\\ 0, & j\in I^c.\end{cases} \tag{3.2}$$

If $\operatorname{card}(I) < \min\{l, n\}$, then by Remark 3.2 the system of equations

$$f_{j,\tau}(x) = 0,\ j \in I, \text{ has no solutions in } \left(K^{\times}\right)^{n}.$$

Hence by using (3.2) the system of equations

$$f_{j,a}(x) = 0,\ j = 1, \dots, l, \text{ has no solutions in } \left(K^{\times}\right)^{n},$$

and so the condition on a in Definition 3.5 is satisfied.

Now, we may assume that $\operatorname{card}(I) \geq \min\{l, n\}$, and that

$$f_{j,\tau}(x) = 0,\ j \in I, \text{ has solutions in } \left(K^{\times}\right)^{n}.$$

Since $\boldsymbol{f}$ is strongly non-degenerate with respect to $\Gamma(\boldsymbol{f})$, it follows that

$$rank_K \left[\frac{\partial f_{j,\tau}}{\partial x_i}(z)\right] = rank_K \left[\frac{\partial f_{j,\tau}}{\partial x_i}(z)\right]_{\substack{j\in I\\ 1\leq i\leq n}} = \min\{l, n\},$$

for any $z \in \left\{z \in (K^{\times})^{n} \mid f_{j,\tau}(z) = 0,\ j \in I\right\}$. Then by (3.2),

$$rank_K \left[\frac{\partial f_{j,a}}{\partial x_i}(z)\right]_{\substack{j\in I\\ 1\leq i\leq n}} = rank_K \left[\frac{\partial f_{j,\tau}}{\partial x_i}(z)\right]_{\substack{j\in I\\ 1\leq i\leq n}} = \min\{l, n\},$$

for any z in

$$\left\{z \in \left(K^{\times}\right)^{n} \mid f_{j,a}(z) = 0,\ j \in I\right\} \supseteq \left\{z \in \left(K^{\times}\right)^{n} \mid f_{j,a}(z) = 0,\ j = 1, \dots, l\right\}.$$

Therefore, $\boldsymbol{f}$ is non-degenerate in the sense of Khovanskii. ∎

Example 3.8. Let $\boldsymbol{f}(x, y) = \left(x^2 - y^2, x^n, y^m\right)$, with $n, m \geq 3$. Then $\boldsymbol{f}$ is not strongly non-degenerate at the origin with respect to $\Gamma(\boldsymbol{f})$. Indeed, $\Gamma(\boldsymbol{f})$ has only one compact facet, τ, that is the straight segment from $(0, 2)$ to $(2, 0)$. Then

$$\boldsymbol{f}_{\tau}(x, y) = \left(x^2 - y^2, 0, 0\right), \text{ and } rank_K \begin{bmatrix} 2z_1 & -2z_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = 1 \neq \min\{2, 3\},$$

for every $(z_1, z_2) \in \left\{(z_1, z_2) \in (K^{\times})^2 \mid z_1^2 - z_2^2 = 0\right\}$, and therefore $\boldsymbol{f}$ is not strongly non-degenerate with respect to $\Gamma(\boldsymbol{f})$. On the other hand, $\boldsymbol{f}$ is non-degenerate in the sense of Khovanskii.

3.3. **Newton polyhedra and log-principalizations.**

Proposition 3.9. *Let $\boldsymbol{f} = (f_1, \dots, f_l) : U(\subseteq K^n) \longrightarrow K^l$ be a polynomial mapping (or more generally, an analytic mapping defined on U) strongly non-degenerate at the origin with respect to $\Gamma(\boldsymbol{f})$. Let $\mathcal{F}_{\boldsymbol{f}}$ be a simple fan subordinated to $\Gamma(\boldsymbol{f})$. Let Y_K be the toric manifold corresponding to $\mathcal{F}_{\boldsymbol{f}}$, and let*

$$\sigma_0 : Y_K \longrightarrow U$$

be the restriction of the corresponding toric map to the inverse image of U. Denote by Z the set of common zeroes of $\mathcal{I}_{\boldsymbol{f}} = (f_1, \dots, f_l)$ in $U \cap (K^{\times})^n$. When U is taken small enough, either $Z = \varnothing$ or it is a submanifold of codimension l. In this last case we have $l < n$ and we denote the closure of Z in U and Y_K by Z_U and Z_Y, respectively.

(1) If $Z = \varnothing$ (or if $l = 1$), the ideal $\sigma_0^*(\mathcal{I}_{\boldsymbol{f}})$ is principal (and monomial) in a sufficiently small neighborhood of $\sigma_0^{-1}\{0\}$.

(2) If $Z \neq \varnothing$, we have that Z_Y is a closed submanifold of Y_K, having normal crossings with the exceptional divisor of σ_0. Let $\sigma_1 : X_K \longrightarrow Y_K$ be the blowing-up of Y_K with center Z_Y, and let $\sigma = \sigma_0 \circ \sigma_1 : X_K \longrightarrow U$. Then the ideal $\sigma^*(\mathcal{I}_{\boldsymbol{f}})$ is principal (and monomial) in a sufficiently small neighborhood of $\sigma^{-1}\{0\}$.

Proof. We first recall the construction of (Y_K, σ_0) from a simple fan $\mathcal{F}_{\boldsymbol{f}}$ subordinated to $\Gamma(\boldsymbol{f})$ (see e.g. [2]). Let Δ_τ be an $n-$dimensional simple cone in $\mathcal{F}_{\boldsymbol{f}}$ such that $F(a) = \tau$ for any $a \in \Delta_\tau$. Then the face τ of $\Gamma(\boldsymbol{f})$ is necessarily a point. Let $a_1, \ldots, a_n$ be the generators of Δ_τ. Then in the chart of Y_K corresponding to Δ_τ, the map σ_0 has the form

$$\begin{array}{cccc} \sigma_0 : & K^n & \longrightarrow & U \\ & y & \longrightarrow & x, \end{array} \tag{3.3}$$

where $x_i = \prod_j y_j^{a_{i,j}}$, with $[a_{i,j}] = [a_1, \ldots, a_n]$. Denote this chart by V_τ. We slightly abuse notation here : since σ_0 only maps to U instead of to the whole of K^n, at some charts it will not be defined everywhere on K^n. If $f_i(x) = \sum_m c_{m,i} x^m$ for $i = 1, \ldots, l$, then

$$(f_i \circ \sigma_0)(y) = \sum_m c_{m,i} \prod_{j=1}^n y_j^{\langle m, a_j\rangle} \text{ for } i = 1, \ldots, l.$$

If $\mathrm{supp}(f_i) \cap \tau \neq \varnothing$, then the minimum of all $\langle m, a_j\rangle$ is attained at τ, and then

$$(f_i \circ \sigma_0)(y) = \left(\prod_{j=1}^n y_j^{d(a_j)}\right) \widetilde{f}_i(y), \text{ with } \widetilde{f}_i(0) \neq 0 \tag{3.4}$$

(cf. [2, page 201, Lemma 8]). If $\mathrm{supp}(f_i) \cap \tau = \varnothing$,

$$(f_i \circ \sigma_0)(y) = \left(\prod_{j=1}^n y_j^{d(a_j)}\right) \widetilde{f}_i(y), \text{ with } \widetilde{f}_i(0) = 0. \tag{3.5}$$

Then, from (3.4) and (3.5), we have in a neighborhood of the origin of V_τ that $\sigma_0^*(\mathcal{I}_{\boldsymbol{f}})$ is generated by $\prod_{j=1}^n y_j^{d(a_j)}$.

Now let us consider on V_τ the points on $\sigma_0^{-1}(0)$, different from the origin of V_τ. We will study simultaneously points with exactly r zero coordinates (where $1 \leq r \leq n-1$); after permuting indices, we may assume that the first r coordinates are zero.

Let τ' be the first meet locus of the cone $\Delta_{\tau'}$ spanned by $a_1, \ldots, a_r$; it is a compact face of $\Gamma(\boldsymbol{f})$ (cf. [2, page 201, Lemma 8]). We can write $(f_i \circ \sigma_0)(y)$ as

$$(f_i \circ \sigma_0)(y) = \left(\prod_{j=1}^r y_j^{d(a_j)}\right) \left(\widetilde{f}_i(y_{r+1}, \ldots, y_n) + O_i(y_1, \ldots, y_n)\right), \tag{3.6}$$

where the $\widetilde{f}_i$ are polynomials in $y_{r+1}, \ldots, y_n$, and the $O_i(y_1, \ldots, y_n)$ are analytic functions in $y_1, \ldots, y_n$ but belonging to the ideal generated by $y_1, \ldots, y_r$. Here the $\widetilde{f}_i$ are identically zero if and only if $\mathrm{supp}(f_i) \cap \tau' = \varnothing$. Furthermore,

$$(3.7) \qquad \left(f_{i,\tau'} \circ \sigma_0\right)(y) = \left(\prod_{j=1}^{r} y_j^{d(a_j)}\right) \widetilde{f}_i\left(y_{r+1}, \ldots, y_n\right).$$

We investigate the $\left(f_i \circ \sigma_0\right)(y)$ for $p = (0, \ldots, 0, p_{r+1}, \ldots, p_n)$ with

$$\widetilde{p} = (p_{r+1}, \ldots, p_n) \in (K^{\times})^{n-r}.$$

We have to study two cases. The first case occurs when there exists an index i such that $\widetilde{f}_i(\widetilde{p}) \neq 0$. In this case as before $\sigma_0^*\left(\mathcal{I}_{\boldsymbol{f}}\right)$ is generated by $\prod_{j=1}^{r} y_j^{d(a_j)}$ in a neighborhood of p.

The second case occurs when $\widetilde{f}_i(\widetilde{p}) = 0$, for all $i = 1, \ldots, l$. We recall that, by the non-degeneracy condition, $rank_K \left[\frac{\partial f_{i,\tau'}}{\partial x_j}(x)\right] = \min\{l, n\}$ for $x \in (K^{\times})^n \cap \{f_{1,\tau'}(x) = \cdots = f_{l,\tau'}(x) = 0\}$. Since σ_0 is an isomorphism over $\left(K^{\times}\right)^n$, then also $rank_K \left[\frac{\partial f_{i,\tau'} \circ \sigma_0}{\partial y_j}(y)\right] = \min\{l, n\}$ for $y \in (K^{\times})^n \cap \{f_{1,\tau'}(\sigma_0(y)) = \cdots = f_{l,\tau'}(\sigma_0(y)) = 0\}$. Note that by (3.7) this condition on y is equivalent to $y \in (K^{\times})^n \cap \{\widetilde{f}_1(y) = \cdots = \widetilde{f}_l(y) = 0\}$ and that $\left[\frac{\partial f_{i,\tau'} \circ \sigma_0}{\partial y_j}(y)\right]$ for such y is equal to

$$\begin{pmatrix} 0 & \ldots & 0 & \left(\prod_{j=1}^{r} y_j^{d(a_j)}\right) \frac{\partial \widetilde{f}_1}{\partial y_{r+1}}(y) & \ldots & \left(\prod_{j=1}^{r} y_j^{d(a_j)}\right) \frac{\partial \widetilde{f}_1}{\partial y_n}(y) \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & \left(\prod_{j=1}^{r} y_j^{d(a_j)}\right) \frac{\partial \widetilde{f}_l}{\partial y_{r+1}}(y) & \ldots & \left(\prod_{j=1}^{r} y_j^{d(a_j)}\right) \frac{\partial \widetilde{f}_l}{\partial y_n}(y) \end{pmatrix}.$$

Now this implies that for $\widetilde{y} = (y_{r+1}, \ldots, y_n) \in (K^{\times})^{n-r} \cap \{\widetilde{f}_1(\widetilde{y}) = \cdots = \widetilde{f}_l(\widetilde{y}) = 0\}$ the rank of the matrix

$$\begin{pmatrix} \frac{\partial \widetilde{f}_1}{\partial y_{r+1}}(\widetilde{y}) & \ldots & \frac{\partial \widetilde{f}_1}{\partial y_n}(\widetilde{y}) \\ \ldots & \ldots & \ldots \\ \frac{\partial \widetilde{f}_l}{\partial y_{r+1}}(\widetilde{y}) & \ldots & \frac{\partial \widetilde{f}_l}{\partial y_n}(\widetilde{y}) \end{pmatrix}$$

is equal to $\min\{l, n\}$. Then necessarily the rank is l, and we must have that $l \leq n - r$.

So when p above satisfies $\widetilde{f}_i(\widetilde{p}) = 0$ for $i = 1, \ldots, l$, then necessarily *all* $\widetilde{f}_i$ are nonzero polynomials, $r \leq n - l$, and $rank_K \left[\frac{\partial \widetilde{f}_i}{\partial y_j}(\widetilde{p})\right] = l$. Now $\left[\frac{\partial \widetilde{f}_i}{\partial y_j}(\widetilde{p})\right] = \left[\frac{\partial (\widetilde{f}_i + O_i)}{\partial y_j}(p)\right]$ (cf. (3.6)). This last matrix having rank l implies that we can choose new coordinates $y' = (y_1, \ldots, y_r, y'_{r+1}, \ldots, y'_n)$ in a neighborhood V_p of p such that

$$(3.8) \qquad \left(f_i \circ \sigma_0\right)(y') = \left(\prod_{j=1}^{r} y_j^{d(a_j)}\right) y'_{r+i} \text{ for } i = 1, \ldots, l.$$

Since σ_0 is an isomorphism on $\left(K^{\times}\right)^n$, we have that $\{y'_{r+1} = \cdots = y'_{r+l} = 0\}$ is the description in V_p of $Z_Y \subset Y$. (The local description (3.8) yields that Z is

a submanifold of $(K^{\times})^n$ of codimension l.) Clearly Z_Y is a submanifold of Y of codimension l, having normal crossings with the exceptional divisor of σ_0.

So, σ_1 being the blowing-up of Y in Z_Y, we obtain by (3.8) that $(\sigma_0 \circ \sigma_1)^*(\mathcal{I}_{\boldsymbol{f}})$ becomes principal. ∎

Remark 3.10. If we replace in Proposition 3.9 the condition *strongly non-degenerate at the origin with respect to* $\Gamma(\boldsymbol{f})$ by the condition *strongly non-degenerate with respect to* $\Gamma(\boldsymbol{f})$, and U by K^n, with a similar proof we obtain a *global* version of the proposition, that is, the conclusions (1) and (2) are valid without the condition *in a sufficiently small neighborhood*. In this case Z_Y may have components that are disjoint with the exceptional divisor of σ_0.

Given $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{N}^n \setminus \{0\}$, we put $\sigma(\xi) := \xi_1 + \ldots + \xi_n$ and $d(\xi) = \min_{x \in \Gamma(\boldsymbol{f})} \langle \xi, x \rangle$ as before. We say that ξ is a primitive vector, if $\gcd(\xi_1, \ldots, \xi_n) = 1$. If $d(\xi) \neq 0$, we define

$$\mathcal{P}(\xi) = \left\{ -\frac{\sigma(\xi)}{d(\xi)} + \frac{2\pi\sqrt{-1}k}{d(\xi)\log q}, \quad k \in \mathbb{Z} \right\}.$$

Let $\mathcal{F}_{\boldsymbol{f}}$ be a simple fan subordinated to $\Gamma(\boldsymbol{f})$. Then the set of generators of the cones in $\mathcal{F}_{\boldsymbol{f}}$, i.e. the skeleton of $\mathcal{F}_{\boldsymbol{f}}$, can be partitioned as $\Lambda_{\boldsymbol{f}} \cup \mathfrak{D}(\Gamma(\boldsymbol{f}))$, where $\Lambda_{\boldsymbol{f}}$ is a finite set of primitive vectors, corresponding to the extra rays, induced by the subdivision into simple cones.

The numerical data of the log-principalizations constructed in Proposition 3.9 and Remark 3.10 can be computed directly from the explicit expressions for the generators of $\sigma_0^*(I_f)$, $\sigma^*(I_f)$, and Lemma 8 in [2, page 201]. Then Theorem 2.4 yields that the poles of $Z_\Phi(s, \boldsymbol{f})$ belong to the set

$$\bigcup_{\xi \in \Lambda_{\boldsymbol{f}}} \mathcal{P}(\xi) \cup \bigcup_{\xi \in \mathfrak{D}(\Gamma(\boldsymbol{f}))} \mathcal{P}(\xi) \cup \left\{ -l + \frac{2\pi\sqrt{-1}k}{\log q}, \, k \in \mathbb{Z} \right\}, \tag{3.9}$$

where the last set may be discarded if $l \geq n$.

This provides a generalization to the case $l \geq 1$ of a well-known result that describes the poles of the local zeta function associated to a non-degenerate polynomial in terms of the corresponding Newton polyhedron [21], [5], [7], [36]. This result was originally established by Varchenko [31] for local zeta functions over $\mathbb{R}$. As in the case $l = 1$, the list (3.9) is too big. More precisely, the set $\cup_{\xi \in \Lambda_{\boldsymbol{f}}} \mathcal{P}(\xi)$ is not necessary. This fact is established by analogous arguments as in [5] where the case $l = 1$ is studied.

Theorem 3.11. *(1) Let $\boldsymbol{f} = (f_1, \ldots, f_l) : U \longrightarrow K^l$ be an analytic mapping strongly non-degenerate at the origin with respect to $\Gamma(\boldsymbol{f})$. If U is a sufficiently small neighborhood of the origin, and Φ is a Schwartz-Bruhat function whose support is contained in U, then the poles of $Z_\Phi(s, \boldsymbol{f})$ belong to the set $\cup_{\xi \in \mathfrak{D}(\Gamma(\boldsymbol{f}))} \mathcal{P}(\xi) \cup \left\{ -l + \frac{2\pi\sqrt{-1}k}{\log q}, \, k \in \mathbb{Z} \right\}$, where the last set may be discarded if $l \geq n$.*
(2) If $\boldsymbol{f} : K^n \longrightarrow K^l$ is a strongly non-degenerate polynomial mapping with respect to $\Gamma(\boldsymbol{f})$, then the poles of $Z(s, \boldsymbol{f})$ belong to the set

$$\cup_{\xi \in \mathfrak{D}(\Gamma(\boldsymbol{f}))} \mathcal{P}(\xi) \cup \left\{ -l + \frac{2\pi\sqrt{-1}k}{\log q}, \, k \in \mathbb{Z} \right\}.$$

The above result can be restated in a geometric form as follows. If s is a pole of $Z_\Phi(s, \boldsymbol{f})$, then $\mathrm{Re}(s)$ is $-l$, or $\mathrm{Re}(s)$ is of the form $-1/t_0$, where $(t_0, \ldots, t_0)$

is the intersection point of the diagonal $\{(t,\ldots,t)\in\mathbb{R}^n\}$ with the supporting hyperplane of a facet of $\Gamma(\boldsymbol{f})$.

By using Theorems 2.7 and 3.11 we obtain the following corollary.

Corollary 3.12. *(1) Let U be a sufficiently small neighborhood of the origin, and let $\boldsymbol{f}=(f_1,\ldots,f_l):U\longrightarrow K^l$ be an analytic mapping strongly non-degenerate at the origin with respect to $\Gamma(\boldsymbol{f})$. Let $(t_{\boldsymbol{f}},\ldots,t_{\boldsymbol{f}})\in\mathbb{Q}^n$ be the intersection point of the diagonal $\{(t,\ldots,t)\in\mathbb{R}^n\}$ with the boundary of $\Gamma(\mathbf{f})$. If $t_{\boldsymbol{f}}\geq 1/l$, then $-1/t_{\boldsymbol{f}}$ is the largest real part of a pole of $Z_U(s,\boldsymbol{f})$.*
(2) Let $\boldsymbol{f}:K^n\longrightarrow K^l$ be a strongly non-degenerate polynomial mapping with respect to $\Gamma(\boldsymbol{f})$. If $t_{\boldsymbol{f}}\geq 1/l$, then $-1/t_{\boldsymbol{f}}$ is the largest real part of a pole of $Z(s,\boldsymbol{f})$.

The largest real part of the poles of $Z(s,\boldsymbol{f})$, $l=1$, when $\boldsymbol{f}$ is non-degenerate with respect to its Newton polyhedron $\Gamma(\boldsymbol{f})$ and $t_{\boldsymbol{f}}>1$ follows from observations made by Varchenko in [31] and was originally noted in the $p-$adic case in [21]. The case $t_{\boldsymbol{f}}=1$ is treated in [7]. The case of $t_{\boldsymbol{f}}<1$ is more difficult and is established in [7] with some additional conditions on $\Gamma(\boldsymbol{f})$ by using a difficult result on exponential sums. In [36] the second author established the case $t_{\boldsymbol{f}}\geq 1$ when $\boldsymbol{f}$ is a non-degenerate polynomial with coefficients in a non-archimedean local field of arbitrary characteristic.

4. Explicit formulas and Newton polyhedra

In [7, Theorem 4.2] Denef and Hoornaert gave an explicit formula for $Z(s,\boldsymbol{f})$, $l=1$, associated to a polynomial $\boldsymbol{f}$ in several variables over the $p-$adic numbers, when $\boldsymbol{f}$ is sufficiently non-degenerate with respect to its Newton polyhedron $\Gamma(\boldsymbol{f})$. This explicit formula can be generalized to the case $l\geq 1$ by using the condition of non-degeneracy for polynomial mappings introduced in this paper.

Let as before K be a $p-$adic field with valuation ring R_K, maximal ideal P_K and residue field $\overline{K}=\mathbb{F}_q$. For any polynomial g over R_K we denote by $\overline{g}$ the polynomial over $\overline{K}$ obtained by reducing each coefficient of g modulo P_K.

Definition 4.1. Let $f_i\in R_K[x]$, $x=(x_1,\ldots,x_n)$, satisfying $f_i(0)=0$ for $i=1,\ldots,l$. The mapping $\boldsymbol{f}=(f_1,\ldots,f_l):K^n\longrightarrow K^l$ is called *strongly non-degenerate over $\overline{K}$ with respect to* $\Gamma(\boldsymbol{f})$, if for any face τ of $\Gamma(\boldsymbol{f})$, including $\Gamma(\boldsymbol{f})$ itself, we have that $rank_K\left[\frac{\partial\overline{f_{i,\tau}}}{\partial x_j}(\overline{z})\right]=\min\{l,n\}$, for any $\overline{z}\in\left(\overline{K}^{\times}\right)^n$ satisfying $\overline{f_{1,\tau}}(\overline{z})=\ldots=\overline{f_{l,\tau}}(\overline{z})=0$. Analogously we call $\boldsymbol{f}$ *strongly non-degenerate at the origin over $\overline{K}$ with respect to* $\Gamma(\boldsymbol{f})$, if the same condition is satisfied but only for the *compact* faces τ of $\Gamma(\boldsymbol{f})$.

Theorem 4.2. *(1) Let $\boldsymbol{f}=(f_1,\ldots,f_l):K^n\to K^l$ be a strongly non-degenerate polynomial mapping over $\overline{K}$. Denote for each face τ of $\Gamma(\boldsymbol{f})$, including $\Gamma(\boldsymbol{f})$ itself,*

$$\overline{D}_\tau:=\left\{\overline{x}\in\left(\overline{K}^{\times}\right)^n\mid\overline{f_{1,\tau}}(\overline{x})=\ldots=\overline{f_{l,\tau}}(\overline{x})=0\right\}.$$

Fix a rational simplicial polyhedral subdivision $\{\Delta_{\tau,i}\}$, with τ a proper face, subordinated to $\Gamma(\boldsymbol{f})$ as in (3.1). Denote by a_j, $j=1,\ldots,r_{\Delta_{\tau,i}}$, the generators of the cone $\Delta_{\tau,i}$. Then

$$Z(s,f)=L_{\Gamma(\boldsymbol{f})}\left(q^{-s}\right)+\sum_{\tau\neq\Gamma(\boldsymbol{f})}L_\tau\left(q^{-s}\right)\left(\sum_i S_{\tau,i}\left(q^{-s}\right)\right).$$

Here

$$L_{\tau}\left(q^{-s}\right)=q^{-n}\left((q-1)^{n}-\frac{card(\overline{D}_{\tau})\left(1-q^{-s}\right)}{1-q^{-\min\{l,n\}-s}}\right),$$

for each face τ *of* $\Gamma\left(\boldsymbol{f}\right)$, *including* $\Gamma\left(\boldsymbol{f}\right)$, *and*

$$S_{\tau,i}\left(q^{-s}\right)=\frac{\left(\sum_{h}q^{\sigma(h)+d(h)s}\right)q^{-\sum_{j=1}^{r_{\Delta_{\tau,i}}}\left(\sigma(a_j)+d(a_j)s\right)}}{\prod_{j=1}^{r_{\Delta_{\tau,i}}}\left(1-q^{-\sigma(a_j)-d(a_j)s}\right)},$$

where h *runs through the elements of the set*

$$\mathbb{Z}^{n}\cap\left\{\sum_{j=1}^{r_{\Delta_{\tau,i}}}\lambda_j a_j\mid 0\leq\lambda_j<1\text{ for }j=1,\ldots,r_{\Delta_{\tau,i}}\right\}.$$

(2) With the same notations and only assuming that $\boldsymbol{f}$ *is strongly non-degenerate at the origin over* $\overline{K}$ *we have*

$$Z_{0}\left(s,\boldsymbol{f}\right)=\sum_{\tau\ compact}L_{\tau}\left(q^{-s}\right)\left(\sum_{i}S_{\tau,i}\left(q^{-s}\right)\right).$$

The proof of the above result is analogous to the case $l=1$ treated in [7, Theorem 4.2].

By using a simple polyhedral subdivision one obtains a slightly less complicated explicit formula in which all the terms $\sum_h q^{\sigma(h)+d(h)s}$ are identically 1. But then in general we have to introduce new rays which give rise to superfluous candidate poles.

Example 4.3. Let $\boldsymbol{f}=\left(x^3-xy,y\right)$ as in Example 2.14. It is strongly non-degenerate over $\overline{K}$ with respect to $\Gamma(\boldsymbol{f})$. We shall compute $Z(s,\boldsymbol{f})$ using Theorem 4.2 and the obvious rational simplicial polyhedral subdivision of $\mathbb{R}_{+}^{2}$. More precisely, set $a_1=(0,1)$, $a_2=(1,3)$, and $a_3=(1,0)$; $\Delta_i=\{a_i\lambda\mid\lambda>0\}$ for $i=1$, 2, 3, and $\Delta_{i,i+1}=\left\{\lambda a_i+\lambda' a_{i+1}\mid\lambda,\lambda'>0\right\}$, $i=1$, 2. Then

$$\mathbb{R}_{+}^{2}=\{0\}\cup\Delta_1\cup\Delta_{1,2}\cup\Delta_2\cup\Delta_{2,3}\cup\Delta_3.$$

With the notation of Theorem 4.2 one easily verifies that all $\overline{D}_{\tau}=\varnothing$ and hence all $L_{\tau}=q^{-2}\left(q-1\right)^{2}$. Further

$$S_{\tau_1}=S_{\tau_3}=\frac{q^{-1}}{1-q^{-1}},\ \ S_{\tau_2}=\frac{q^{-4-3s}}{1-q^{-4-3s}},$$

$$S_{\tau_{1,2}}=\frac{q^{-5-3s}}{\left(1-q^{-1}\right)\left(1-q^{-4-3s}\right)},\ \ S_{\tau_{2,3}}=\frac{\left(1+q^{2+s}+q^{3+2s}\right)q^{-5-3s}}{\left(1-q^{-1}\right)\left(1-q^{-4-3s}\right)}.$$

Therefore

$$Z(s,\boldsymbol{f})=q^{-2}\left(q-1\right)\frac{\left(q+1+q^{-1-s}+q^{-2-2s}\right)}{1-q^{-4-3s}}.$$

If we would use the natural *simple* polyhedral subdivision of the one above, introducing two new rays generated by $(1,1)$ and $(1,2)$, we would introduce the same superfluous (real) candidate poles -2 and $-\frac{3}{2}$ as in Example 2.14. This is reasonable because the log-principalization of Proposition 3.9 associated to this simple fan is in fact the same as the one constructed in Example 2.14.

Example 4.4. Let $\boldsymbol{f} = \left(y^2 - x^3, y^2 - z^2\right)$ as in Example 2.15. When $char\left(\overline{K}\right) \neq 2$, it is strongly non-degenerate at the origin over $\overline{K}$ with respect to $\Gamma(\boldsymbol{f})$. The Newton polyhedron $\Gamma(\boldsymbol{f})$ has seven compact faces. The polyhedral subdivision associated to it is already simplicial, so in the formulation of Theorem 4.2 (2) we need to sum over seven cones: the ray through $a = (2, 3, 3)$, the three $2-$dimensional cones with a in their boundaries, and the three $3-$dimensional cones. We note that all the $\overline{D}_\tau = \varnothing$, except when τ is the unique compact facet, in this case $\mathrm{card}\left(\overline{D}_\tau\right) = 2\,(q-1)$. Concerning the $S_\tau\left(q^{-s}\right)$ we just mention that the expression $\sum_h q^{\sigma(h)+d(h)s}$ is three times equal to 1, three times equal to $1+q^{3+2s}+q^{6+4s}$, and once to $1+q^{5+3s}$. One can verify that the formula in Theorem 4.2 yields the same expression for $Z_0(s, \boldsymbol{f})$ as in Example 2.15. Note that $-8/6$ and -2 are the only (real) candidate poles given by Theorems 3.11 or 4.2.

Remark 4.5. With the obvious analogous definitions for strongly non-degeneracy over $\mathbb{C}$, we have the following. Suppose that $f_1, \ldots f_l$ are polynomials in n variables with coefficients in a number field F ($\subseteq \mathbb{C}$). Then we can consider $\boldsymbol{f} = (f_1, \ldots, f_l)$ as a map $K^n \to K^l$ for any non-archimedean completion K of F. If $\boldsymbol{f}$ is strongly non-degenerate at the origin over $\mathbb{C}$ with respect to $\Gamma(\boldsymbol{f})$, then $\boldsymbol{f}$ is strongly non-degenerate over $\overline{K}$ with respect to $\Gamma(\boldsymbol{f})$ for almost all the completions K of F. (And analogously for non-degeneracy at the origin.) This fact follows by applying the Weak Nullstellensatz.

Remark 4.6. By using our notion of non-degeneracy with respect to a Newton polyhedron it is also possible to give lists of candidate poles and explicit formulae for the motivic and topological zeta functions introduced in 2.4, associated to a polynomial ideal. These explicit formulas are reasonably straightforward generalizations of those in [1] and [10, Théorème 5.3 (i)]. For the topological zeta function one requires here strongly non-degeneracy with respect to all the faces of the "global" Newton polyhedron as in [10, (5.1)].

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